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Superlinear Indefinite Elliptic Problems and Pohožaev Type Identities

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We prove the existence of a nontrivial solution for a nonlinear elliptic problem $-\Delta u = \mu u + a(x)\, g(u)$ with Dirichlet boundary condition on a bounded domain, where g is superlinear both at zero and at infinity, $a(x)$ changes sign and $\mu > 0$.

Key Words: superlinear elliptic problems; minimax methods; Morse theory.

INTRODUCTION

In this paper we seek nonzero solutions for

$$-\Delta u = \mu u + a(x)\, g(u), \qquad u \in H_0^1(\Omega), \tag{$(P)_\mu$}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geqslant 2)$ is a bounded, connected open set with $C^{1,1}$ boundary, $a \in C^2(\bar{\Omega})$ changes sign in Ω, $\mu > 0$ is a real parameter and $g \in C^1(\mathbb{R}; \mathbb{R})$ has a superlinear behavior both at zero and at infinity.

For small values of μ, the existence of positive solutions for $(P)_\mu$ was proved by several authors, see, e.g., [1, 2, 5, 7, 8] and their references. The

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existence of infinitely many (possibly sign changing) solutions was proved in [1] for odd functions g and in [4] for a class of perturbations of odd functions, for (nearly) every μ. In [1, 3, 5] the authors exhibit numbers $\bar{\mu}>0$ depending on a such that the problem has several nonzero solutions for $\mu<\bar{\mu}$. We also mention that the corresponding ordinary differential equations with periodic boundary condition was studied in [9].

Here we let $\mu>0$ in $(P)_\mu$ be arbitrarily large. In order to motivate our result, denote

$$\Omega^- := \{x \in \Omega : a(x)<0\} \quad \text{and} \quad \Omega^+ := \{x \in \Omega : a(x)>0\}.$$

In [1, 3, 4] it is assumed that a has a "thick" zero set, namely that $\overline{\Omega^+} \cap \overline{\Omega^-} = \emptyset$. That assumption is crucial in the arguments employed there in order to obtain the Palais–Smale compactness condition. Here we concentrate on the case where the zero set,

$$\Omega^0 := \{x \in \bar{\Omega} : a(x)=0\},$$

has Lebesgue measure zero (this follows from assumption (H2) below). Our interest is motivated by the result of [7], where the case $\mu<\mu_1$ is treated, and by an example in [3] where the authors exhibit a sequence of functions a_n and numbers μ_n^* for which, under some assumptions on g, problem $(P)_\mu$ has a nonzero solution for $\mu<\mu_n^*$. We can see in that example that the measure of the zero set of a_n tend to zero as $n\to\infty$ and also that $\mu_n^*\to\infty$. This suggests our Theorem 1 below.

We also mention that, contrarily to [1, 3, 4] and similarly to [7], we do not assume g to be a quadratic perturbation of a power function $|u|^{p-2}u$. In particular we allow quadratic perturbations of functions $|u|^{p-2}u+|u|^{q-2}u$ with $2<p$, $q<2^*:=2N/(N-2)$. As explained in [1, 3], this creates difficulties in establishing the required compacity for problem $(P)_\mu$, see Section 1.

Let $(\mu_i)_{i\geqslant 1}$ be the increasing sequence of the eigenvalues of $(-\Delta, H_0^1(\Omega))$, $\nu(x)$ the unit outward normal of Ω at the point $x\in\partial\Omega$ and denote by $(\cdot,\cdot)$ the inner product in $\mathbb{R}^N$. We prove the following.

THEOREM 1. *Assume* $\Omega^+\neq\emptyset$, $\Omega^-\neq\emptyset$, $\partial\Omega^-$ *is locally Lipschitz and, for some* $\varepsilon>0$, $\ell>0$, $2<p<2^*$, $k\in\mathbb{N}$,

(H1) $(\nabla a(x), \nu(x))\leqslant 0$ *for all* $x\in\partial\Omega$ *such that* $a(x)\leqslant 0$ *and* $\operatorname{dist}(x,\Omega^0)<\varepsilon$;

(H2) $\nabla a(x)\neq 0$, *for all* $x\in\Omega^0$;

(H3) $\mu_k<\mu<\mu_{k+1}$;

(H4) $g'(0) = g(0) = 0$;

(H5) $\lim_{|u| \to \infty} g'(u)/(p-1)\,|u|^{p-2} = \ell$.

Then problem $(P)_\mu$ *has a nonzero solution.*

We point out that (H1) holds trivially in the situation where a does not vanish on $\partial\Omega$. This includes the quoted example in [3]. That situation, together with assumptions (H2), (H4) and (H5) was also considered in [7] in finding positive solutions for $(P)_\mu$ with $\mu < \mu_1$. In fact in [7] condition (H5) is replaced with $\lim_{u \to +\infty} g(u)/u^p = \ell > 0$ for $2 < p < (2N+1)/(N-1)$. Observe that here we allow $p < 2N/(N-2)$. On the other hand, (H1) also holds in the simple example where Ω is a ball centered at the origin and a is a linear projection; in this case $(\nabla a(x), \nu(x)) \leqslant 0$ for every $x \in \partial\Omega$ such that $a(x) \leqslant 0$.

The rest of the paper is devoted to the proof of Theorem 1. First, we apply a local version of the linking theorem [14, 15] to a suitable sequence of truncated problems. An inspection of the proof in [14], together with the results in [13], allow us to establish the existence of a sequence of solutions satisfying additional estimates (independent of the truncation) either on the energy level or on their Morse index. The bounded energy case and the bounded Morse index case will require different technical arguments. In both cases we will show that the above mentioned bounds lead to L^∞ estimates. In order to treat the bounded Morse index case we shall combine the blow-up arguments in [6, 7, 10] with a careful analysis of the limiting problems. Similarly to [6], we shall achieve our goal by combining spectral analysis (boundedness of the Morse index) with estimates which are reminiscent of the Pohožaev identity (see, e.g., [19]).

In proving Theorem 1 we were led to the study of nontrivial bounded solutions of elliptic equations in $\mathbb{R}^N$. In particular, in Section 4 we state some nonlinear Liouville type theorems which may be useful in other situations.

1. A MODIFIED PROBLEM

We seek for nonzero critical points in $H := H_0^1(\Omega)$ for the energy functional associated to problem $(P)_\mu$. The norm in H is given by $\|u\| := \|\nabla u\|_{L^2(\Omega)}$.

Since a changes sign, the superquadratic term of the functional is indefinite and it is not clear whether one can split H in linear subspaces in such a way that a minimax procedure applies. In addition, the Palais–Smale condition for the energy functional does not seem to follow readily from our assumptions. In order to overcome these difficulties we study a truncated problem as follows. Without loss of generality we assume $\ell = 1$ in (H5).

Let $a_j \to +\infty$ be any sequence and $q \in]2, p[$ be a fixed number close enough to p so that

$$3\,\frac{q-2}{p-2} > 1, \qquad \frac{p-1}{q-1} > \frac{2}{3}, \qquad \frac{p-q}{q-1} < \frac{1}{3}, \qquad \text{and} \qquad p-q<1. \tag{1}$$

Define

$$g_j(u) := \begin{cases} \tilde{A}_j\,|u|^{q-2}\,u + \tilde{B}_j, & \text{for} \quad u \leqslant -a_j; \\ g(u), & \text{for} \quad |u| \leqslant a_j; \\ A_j\,|u|^{q-2}\,u + B_j, & \text{for} \quad u \geqslant a_j. \end{cases}$$

The coefficients are chosen in such a way that g_j is C^1. We list the relevant properties of g and g_j. The corresponding proofs are given in Section 5. We let $G(u) := \int_0^u g(\xi)\,d\xi$, $G_j(u) := \int_0^u g_j(\xi)\,d\xi$.

(A1) For every $\varepsilon > 0$ there exist $j_0 \in \mathbb{N}$, $C > 0$ such that, for every $j \geqslant j_0$, $u \in \mathbb{R}$,

$$g(u)\,u - qG(u) \geqslant \left(1 - \frac{q}{p} - \varepsilon\right) g(u)\,u - C,$$

$$g_j(u)\,u - qG_j(u) \leqslant \left(1 - \frac{q}{p} + \varepsilon\right) g_j(u)\,u + C.$$

(A2) For every $j \in \mathbb{N}$ and $\theta \in]q, p[$ there exist $\varepsilon > 0$, $C > 0$ such that, for every $u \in \mathbb{R}$,

$$g(u)\,u - \theta G(u) \geqslant \varepsilon\,|u|^p - C,$$

$$\theta G_j(u) - g_j(u)\,u \geqslant \varepsilon\,|u|^q - C.$$

(A3) There exist C_1, C_2 and $j_0 \in \mathbb{N}$ such that, for every $j \geqslant j_0$, $u \in \mathbb{R}$,

$$g_j(u)\,u \geqslant C_1\,|u|^q - C_2.$$

(A4) There exist $A, B, C > 0$ and $j_0 \in \mathbb{N}$ such that, for every $j \geqslant j_0$, $u \in \mathbb{R}$,

$$Ag_j(u)\,u - C \leqslant G_j(u) \leqslant Bg_j(u)\,u + C.$$

Consider the modified problem

$$-\Delta u = \mu u + a^+(x)\,g(u) - a^-(x)\,g_j(u), \qquad u \in H_0^1(\Omega), \qquad (P)_{\mu,\,j}$$

where $a^+ := \max\{a, 0\}$, $a^- := a^+ - a$. The energy functional is given by

$$E_j(u) = \frac{1}{2}\int_\Omega (|\nabla u|^2 - \mu u^2) - \int_\Omega a^+ G(u) + \int_\Omega a^- G_j(u)$$

for $u \in H := H_0^1(\Omega)$. The regularity and the subcritical growth of g imply that $E_j \in C^2(H; \mathbb{R})$, $\nabla E_j - \mathrm{Id}$ is compact and also that $D^2 E_j(u) - \mathrm{Id}$ is compact for every $u \in H$.

A nonzero critical point for E_j is obtained by applying the local linking theorem of Liu and Li [14] (see also [15, Th. 2] or [16, Th. 9.6]). The following auxiliary result is probably known. Since we could not find a precise reference, we present a quick proof pointed us by Luc Tartar, to whom we acknowledge. Let $(\varphi_i)_{i \geqslant 1}$ be the sequence of eigenfunctions associated to the eigenvalues $(\mu_i)_{i \geqslant 1}$.

Lemma 2. *Let $\omega \subset \Omega$ be any open subset of the connected open set Ω and $\alpha_i \in \mathbb{R}$, $i = 1, ..., m$. If $\sum \alpha_i \varphi_i = 0$ in ω then $\alpha_i = 0$ for all $i = 1, ..., m$.*

Proof. (1) Given $\mu \in \mathbb{R}$ and any open balls $B_{r_1}(x_0) \subset B_{r_2}(x_0) \subset \Omega$, the unique continuation property implies that if $-\Delta u = \mu u$ in $B_{r_2}(x_0)$ and $u = 0$ in $B_{r_1}(x_0)$ then $u = 0$ in $B_{r_2}(x_0)$. An elementary connectedness argument then shows that if $-\Delta u = \mu u$ in Ω and $u = 0$ in ω then $u = 0$ in Ω. As a consequence, we may already assume that $\mu_i \neq \mu_j$ for every distinct $i, j \in \{1, ..., m\}$.

(2) Successive applications of the operator $-\Delta$ in the identity of the lemma yield $\sum_i \alpha_i \mu_i^k \varphi_i = 0$ in ω, for every $k = 0, ..., m-1$. In particular, for any given numbers $\beta_0, ..., \beta_{m-1}$,

$$\sum_{i,k} \beta_k \alpha_i \mu_i^k \varphi_i = 0 \qquad \text{in } \omega.$$

The matrix with $A = [a_{ik}]$ with entries $a_{ik} = \mu_i^k$ has a non zero determinant $\prod_{i>j} (\mu_i - \mu_j)$. Therefore, given $j \in \{1, ..., m\}$, we can solve

$$\sum_k \beta_k \mu_i^k = \delta_{ij} \text{ (Kronecker symbol)}, \qquad i = 1, ..., m.$$

Thus $\alpha_j \varphi_j = 0$ in ω and therefore $\alpha_j = 0$. Since j was arbitrary, this proves the lemma. ▮

The existence of a nonzero critical point is a consequence of the following.

PROPOSITION 3. *For each* $j \in \mathbb{N}$,

(a) E_j *has a local linking at the origin,*

(b) $E_j(u) \to -\infty$ *as* $\|u\| \to \infty$ *over any finite dimensional eigenspace,*

(c) E_j *satisfies both* (PS) *and* $(PS)^*$ *conditions over* H.

Proof. For k as in assumption (H3), denote H_1 the eigenspace associated to the eigenvalues $\mu_1, ..., \mu_k$ and let $H_2 := H_1^\perp$. The statement in (a) means that there exists $r > 0$ such that

$$\sup_{\partial B_r(0) \cap H_1} E_j < 0 < \inf_{\partial B_r(0) \cap H_2} E_j.$$

This follows immediately from (H3) and the fact that (H4), (H5) imply that both G and G_j are bounded by $\varepsilon |u|^2 + C_\varepsilon |u|^p$ for arbitrarily small ε.

As for (b) let Y be a subspace of H generated by a finite number of eigenfunctions. Then there exists $\delta > 0$, depending on Y, such that

$$\int_\Omega a^+ |u|^p \geqslant \delta \|u\|^p, \qquad \forall u \in Y. \tag{2}$$

Otherwise, a compactness argument would yield some $u \in Y$ with $\|u\| = 1$ and $\int_\Omega a^+ |u|^p = 0$. This contradicts Lemma 2. Now, taking (2) into account we see that there exist positive constants c_1, c_2, c_3 such that

$$E_j(u) \leqslant \|u\|^2 - c_1 \|u\|^p + c_2 \|u\|^q + c_3, \qquad \forall u \in Y.$$

Since $2 < q < p$, this proves (b).

Concerning statement (c), for each $n \in \mathbb{N}$ denote the eigenspace $Y_n := \mathrm{sp}(\varphi_1, ..., \varphi_n)$. Let $(u_n) \subset H$ be a $(PS)^*$ sequence with respect to (Y_n), i.e.,

$$u_n \in Y_n, \qquad \overline{\lim_{n \to \infty}}\, E_j(u_n) < \infty, \qquad \nabla E_j|_{Y_n}(u_n) \to 0.$$

We must prove that (u_n) has a convergent subsequence to a critical point of E_j. By standard arguments it is in fact sufficient to prove that (u_n) is bounded in H. Assume by contradiction that $t_n := \|u_n\| \to +\infty$. Up to a subsequence, there exists a weak limit v_0 of $v_n := u_n / t_n$. Fix any number $\theta \in]q, p[$. Since $\theta E_j(u_n) - \nabla E_j(u_n)\, u_n \leqslant o(t_n)$,

$$\left(\frac{\theta}{2} - 1\right) t_n^2 + \int_\Omega a^+(g(u_n)\, u_n - \theta G(u_n)) + \int_\Omega a^-(\theta G_j(u_n) - g_j(u_n)\, u_n)$$

$$\leqslant \left(\frac{\theta}{2} - 1\right) \mu \int_\Omega u_n^2 + o(t_n).$$

From property (A2) we deduce that, for some $C>0$,

$$t_n^2+\int_\Omega a^+\,|u_n|^p+\int_\Omega a^-\,|u_n|^q\leqslant C\int_\Omega u_n^2+o(t_n). \tag{3}$$

In particular,

$$\int_\Omega |a|\,|u_n|^q\leqslant C\int_\Omega u_n^2+o(t_n),$$

so that

$$t_n^{q-2}\int_\Omega |a|\,|v_n|^q \qquad \text{is bounded.}$$

Since $q>2$ and $v_n\to v_0$ in $L^q(\Omega)$, we conclude $\int_\Omega |a|\,|v_0|^q=0$. By assumption, $a(x)\neq 0$ a.e. in Ω and so we must have $v_0=0$. In particular, $\int_\Omega v_n^2\to 0$. But (3) implies $1\leqslant C\int_\Omega v_n^2+o(1)$, a contradiction. This proves the $(PS)^*$ condition. A similar (and easier) argument yields the (PS) condition for E_j, namely that any sequence $(u_n)\subset H$ with $\overline{\lim}\,E_j(u_n)<\infty$ and $\lim\|\nabla E_j(u_n)\|=0$ has a convergent subsequence to a critical point of E_j. ∎

According to the quoted local linking theorem, E_j admits a nonzero critical point u_j. Then $u_j\in C^2(\Omega)\cap C^1(\bar{\Omega})$ and is a solution for problem $(P)_{\mu,\,j}$. In the following we may of course assume $a_j\leqslant\|u_j\|_{L^\infty(\Omega)}$, otherwise we are done. An inspection of the proof of that theorem shows that either $E_j(u_j)\leqslant 0$ or else u_j is constructed through a minimax procedure (precisely, through a variant of the well known Rabinowitz's linking theorem). Denote $m_j(u_j)$ the Morse index of u_j with respect to E_j, that is the supremum of the dimensions of the linear subspaces of H on which the quadratic form $D^2E_j(u_j)$ is negative definite. Standard estimates on the Morse index (see, e.g., [13; 17; 16; Th. 10.17]) imply then the following: *up to a subsequence*,

$$\textit{either}\qquad E_j(u_j)\leqslant 0,\quad \forall j,\qquad \textit{or}\qquad m_j(u_j)\leqslant k+1,\quad \forall j.$$

In Section 2 we prove that in the first situation the sequence (u_j) is bounded in $H_0^1(\Omega)$. Elliptic regularity implies then that (u_j) is also bounded in $L^\infty(\Omega)$ and thus u_j is a solution of the original problem for large j. In Section 3 we use a blow-up argument to show that if $(m_j(u_j))$ is bounded then (u_j) is bounded in $L^\infty(\Omega)$ and this will complete the proof of Theorem 1. We note that it is most likely that in fact $m_j(u_j)\leqslant k+1$ in any case, as the proof of the local linking suggests; however, in Section 3 we shall need the main estimates deduces in Section 2.

Our blow-up arguments in Section 3 rely on some new Liouville type theorems. Since these results may be of intrinsic interest, we state them in a separate section (Section 4).

2. THE BOUNDED ENERGY CASE

Let $u_j \in C^2(\Omega) \cap C^1(\bar{\Omega})$ be a sequence of solutions of problems $(P)_{\mu, j}$. Denote $t_j = \|u_j\|$ and $\|u_j\|_2 = \|u_j\|_{L^2(\Omega)}$. We must prove that (t_j) is bounded. Let

$$S_j^+ := \int_\Omega a^+ \frac{g(u_j)\, u_j}{t_j^2}, \qquad S_j^- := \int_\Omega a^- \frac{g_j(u_j)\, u_j}{t_j^2}.$$

Proposition 4. *Under assumptions* (H1), (H2), (H5), (S_j^-) *is bounded.*

Proof. (1) Let $\chi\colon \mathbb{R}^N \to [0, 1]$ be a smooth function. Denote by n the unit outward normal of Ω^-. We compute

$$\begin{aligned} \operatorname{div} U_j &:= \operatorname{div}(a\chi(\nabla u_j, \nabla a)\, \nabla u_j - \tfrac{1}{2} a\chi\, |\nabla u_j|^2\, \nabla a) = a\chi(\nabla u_j, \nabla a)\, \Delta u_j \\ &\quad + \chi(\nabla u_j, \nabla a)^2 + a\chi \sum_{i,k} \frac{\partial^2 a}{\partial x_i\, \partial x_k} \frac{\partial u_j}{\partial x_i} \frac{\partial u_j}{\partial x_k} - \chi\, |\nabla a|^2 \frac{|\nabla u_j|^2}{2} \\ &\quad - a\chi \frac{|\nabla u_j|^2}{2}\, \Delta a + a(\nabla u_j, \nabla a)(\nabla u_j, \nabla \chi) - a \frac{|\nabla u_j|^2}{2} (\nabla \chi, \nabla a). \end{aligned}$$

Integrate over Ω^- to deduce

$$\int_{\Omega^-} a\chi(\nabla u_j, \nabla a)\, \Delta u_j = \int_{\partial\Omega^-} (U_j, n) + O(t_j^2).$$

Similarly,

$$\begin{aligned} \operatorname{div} V_j &:= \operatorname{div}(a^2 \chi G_j(u_j)\, \nabla a) = a^2 \chi g_j(u_j)(\nabla u_j, \nabla a) \\ &\quad + 2a\, |\nabla a|^2\, \chi G_j(u_j) + a^2 G_j(u_j)(\nabla \chi, \nabla a) + a^2 \chi G_j(u_j)\, \Delta a, \end{aligned}$$

so that, since V_j vanishes on $\partial\Omega^-$,

$$\begin{aligned} &\int_{\Omega^-} a^2 \chi g_j(u_j)(\nabla u_j, \nabla a) \\ &\quad = 2 \int_{\Omega^-} a^-\, |\nabla a|^2\, \chi G_j(u_j) - \int_{\Omega^-} a^2 G_j(u_j)(\nabla \chi, \nabla a) - \int_{\Omega^-} a^2 \chi G_j(u_j)\, \Delta a. \end{aligned}$$

Observe also that, according to properties (A3)–(A4), G_j is bounded below, uniformly in j. Multiply the equation in $(P)_{\mu,j}$ by $a\chi(\nabla u_j, \nabla a)$ and use the above identities to deduce

$$2\int_{\Omega^-} a^- |\nabla a|^2 \chi G_j(u_j) \leqslant C\int_{\Omega^-} a^2 G_j(u_j) + O(t_j^2) - \int_{\partial\Omega^-} (U_j, n) \tag{4}$$

for some $C>0$.

(2) Multiply the equation in $(P)_{\mu,j}$ by au_j and integrate over Ω^- to deduce

$$\int_{\Omega^-} a^2 g_j(u_j)\, u_j = O(t_j^2). \tag{5}$$

This, combined with (4) and property (A4) yields

$$\int_{\Omega^-} a^- |\nabla a|^2 \chi g_j(u_j)\, u_j \leqslant O(t_j^2) - D\int_{\partial\Omega^-} (U_j, n) \tag{6}$$

for some $D>0$.

(3) For ε given by (H1), let

$$\Omega_\varepsilon^0 := \{x \in \Omega^- : \operatorname{dist}(x, \Omega^0) < \varepsilon\}.$$

Choose χ as in Step 1 with the additional property that $\chi = 1$ over $\Omega_{\varepsilon/2}^0$ and $\chi(x) = 0$ for every $x \in \Omega^-$ such that $\operatorname{dist}(x, \Omega^0) \geqslant \varepsilon$. We claim that

$$\int_{\partial\Omega^-} (U_j, n) \geqslant 0.$$

Indeed, from the expression of U_j we see that we only have to consider points $x \in \overline{\Omega^-} \cap \partial\Omega$ such that $\operatorname{dist}(x, \Omega^0) \leqslant \varepsilon$. Since u_j vanishes on $\partial\Omega$, we have $\nabla u_j(x) = (\nabla u_j(x), \nu(x))\, \nu(x)$ so that, from (H1),

$$(U_j(x), n(x)) = \tfrac{1}{2} a(x)\, \chi(x)\, |\nabla u_j(x)|^2\, (\nabla a(x), \nu(x)) \geqslant 0,$$

and this proves the claim.

(4) We now use assumption (H2). Since $\inf_{\Omega_\varepsilon^0} |\nabla a| > 0$, it follows from (6) and Step 3 that

$$\int_{\Omega_\varepsilon^0} a^- g_j(u_j)\, u_j = O(t_j^2).$$

But clearly (5) shows that also

$$\int_{\Omega^- \setminus \Omega_\varepsilon^0} a^- g_j(u_j)\, u_j = O(t_j^2)$$

and this gives the conclusion. ∎

Now we can prove that (t_j) is bounded. Indeed, multiply the equation in $(P)_{\mu, j}$ by u_j and integrate over Ω to obtain

$$S_j^+ = S_j^- + 1 - \mu \, \|u_j\|_2^2 / t_j^2. \tag{7}$$

In particular, (S_j^+) is also bounded. Up to subsequences, let $S^\pm := \lim S_j^\pm \in \mathbb{R}$. Multiply the equation in $(P)_{\mu, j}$ by au_j and integrate over Ω to obtain

$$\int_\Omega (a^+)^2\, g(u_j)\, u_j + \int_\Omega (a^-)^2\, g_j(u_j)\, u_j = O(t_j^2).$$

From property (A3) we deduce

$$\int_\Omega a^2\, |u_j|^q = \int_\Omega (a^+)^2\, |u_j|^q + \int_\Omega (a^-)^2\, |u_j|^q = O(t_j^2). \tag{8}$$

Suppose by contradiction that $t_j \to +\infty$. From (8) we deduce (compare with (3) above) that, up to a subsequence, (u_j/t_j) converges weakly to zero. Then (7) implies

$$S^+ = S^- + 1. \tag{9}$$

Now we use the fact that $(E_j(u_j))$ is bounded above. Precisely, since $qE_j(u_j) - \nabla E_j(u_j)\, u_j \leqslant 0$ we have

$$\int_\Omega a^+ (g(u_j)\, u_j - qG(u_j)) \leqslant \int_\Omega a^- (g_j(u_j)\, u_j - qG_j(u_j)) + o(t_j^2).$$

Let $\delta > 0$. From property (A1) we see that, for large j,

$$\left(1 - \frac{q}{p} - \delta\right) S_j^+ \leqslant \left(1 - \frac{q}{p} + \delta\right) S_j^- + o(1).$$

Hence

$$\left(1 - \frac{q}{p} - \delta\right) S^+ \leqslant \left(1 - \frac{q}{p} + \delta\right) S^-, \qquad \forall \delta > 0.$$

Letting $\delta \to 0$ we conclude $S^+ \leqslant S^-$ and this contradicts (9). Thus (t_j) is bounded in case $(E_j(u_j))$ is bounded above as claimed in Section 1.

3. THE BOUNDED MORSE INDEX CASE

Let $u_j \in C^2(\Omega) \cap C^1(\bar{\Omega})$ be a sequence of solutions of problems $(P)_{\mu, j}$. In this section we suppose that $m_j(u_j) \leqslant k+1$ and prove that, at least for a subsequence, $\|u_j\|_\infty := \|u_j\|_{L^\infty(\Omega)}$ is bounded. Assume by contradiction that

$$M_j := \|u_j\|_\infty = \max_\Omega u_j = u_j(x_j) \to \infty$$

for some $x_j \in \Omega$ (the case $M_j = \max_\Omega(-u_j)$ is similar). For a given sequence $\lambda_j \to 0$, define

$$v_j(x) := \frac{u_j(\lambda_j x + x_j)}{M_j}, \qquad x \in \Omega_j := \frac{1}{\lambda_j}(\Omega - x_j).$$

We may assume $x_j \to x_0 \in \bar{\Omega}$. In the sequel we need to consider several cases depending upon the localization of x_0 in $\bar{\Omega}$. Although each of them requires some particular technical argument, the underlying idea will be the same for each case: The sequence of blow-up functions converges to a nonzero function v satisfying a limit boundary value problem, while the boundedness of their Morse indexes yields some integrability properties for v. On the other hand, these latter properties are shown to imply $v = 0$ and this is a contradiction. Thus $(\|u_j\|_\infty)$ is bounded and this ends the proof of Theorem 1.

LEMMA 5. *We have $a(x_0) \geqslant 0$.*

Proof. If $a(x_0) < 0$ then $a^-(x_j) \geqslant \varepsilon$ for some $\varepsilon > 0$ and every j large. Since $\Delta u_j(x_j) \leqslant 0$, the equation in $(P)_{\mu, j}$ together with property (A3) imply the contradiction

$$\mu \geqslant \varepsilon C_1 M_j^{q-2} - \varepsilon C_2 M_j^{-2}. \quad \blacksquare$$

Now, four different cases may occur.

Case A. Suppose $x_0 \in \Omega$ and $a(x_0) > 0$. Let then λ_j be given by

$$\lambda_j^2 M_j^{p-2} = 1.$$

Denoting $a_j^\pm(x) := a_j^\pm(\lambda_j x + x_j)$, we see that v_j satisfies

$$\Delta v_j + \mu \lambda_j^2 v_j + \theta_j(x) = 0, \qquad x \in \Omega_j,$$

where

$$\theta_j(x) := a_j^+(x) \frac{g(M_j v_j)}{M_j^{p-1}} - a_j^-(x) \frac{g_j(M_j v_j)}{M_j^{p-1}}.$$

Since θ_j is uniformly bounded and $x_0 \in \Omega$, elliptic estimates imply that over any ball $B_R(0)$ and up to a subsequence, $v_j \to v$ in $W^{2,r} \cap C^{1,\beta}$ ($r > N$, $0 < \beta < 1$) and $v(0) = 1$ (see [10, p. 889]). Observe that, since $a(x_0) > 0$, $\theta_j(x) \to a(x_0) \, |v(x)|^{p-2} \, v(x)$ for every point $x \in \mathbb{R}^N$. Therefore, by the arguments in [10], v is defined in all of $\mathbb{R}^N$, is C^2 and satisfies

$$\Delta v + a(x_0) \, |v|^{p-2} \, v = 0, \qquad x \in \mathbb{R}^N.$$

For every function $\varphi \in H^1(\mathbb{R}^N)$, let

$$E''(v) \, \varphi, \varphi := \int_{\mathbb{R}^N} |\nabla\varphi|^2 - a(x_0)(p-1) \int_{\mathbb{R}^N} |v|^{p-2} \, \varphi^2. \tag{10}$$

Our next lemma states that v has finite index in the sense of the definition given at the beginning of Section 4.

LEMMA 6. *There exists* $R_0 > 0$ *such that* $E''(v) \, \varphi, \varphi \geqslant 0$ *for every* $\varphi \in H_0^1(\mathbb{R}^N \backslash B_{R_0}(0))$.

Proof. Let $\varphi \in \mathcal{D}(\mathbb{R}^N)$ be such that $E''(v) \, \varphi, \varphi < 0$. The uniform convergence of v_j to v on compact sets implies that

$$I_j := \int_{\mathbb{R}^N} |\nabla\varphi|^2 - \mu\lambda_j^2 \int_{\mathbb{R}^N} \varphi^2 - \int_{\mathbb{R}^N} \left(a_j^+ \frac{g'(M_j v_j)}{M_j^{p-2}} - a_j^- \frac{g_j'(M_j v_j)}{M_j^{p-2}} \right) \varphi^2 < 0$$

for large j. Defining $\chi_j(x) := \varphi((x - x_j)/\lambda_j)$, we see that $\chi_j \in \mathcal{D}(\Omega)$ and

$$D^2 E_j(u_j) \, \chi_j, \chi_j = \lambda_j^{N-2} I_j < 0.$$

Since $(m_j(u_j))$ is bounded, the lemma follows easily. ∎

It follows from Lemma 6 and Proposition 10 in Section 4 that $v = 0$. This contradicts $v(0) = 1$ and ends the proof of Theorem 1 if case A holds.

Case B. Suppose $x_0 \in \partial\Omega$ and $a(x_0) > 0$. Again we take $\lambda_j^2 M_j^{p-2} = 1$. Denote

$$d_j := \text{dist}(x_j, \partial\Omega) = |x_j - z_j| \to 0$$

for some $z_j \in \partial\Omega$. In case $d_j/\lambda_j \to \infty$ we see that, for every $x \in \mathbb{R}^N$, $\lambda_j x + x_j \in \Omega$ for large j, so that we can proceed exactly as in Case A. Suppose now that, for a subsequence,

$$\frac{d_j}{\lambda_j} \to d_0 \in [0, \infty[.$$

For completeness, we sketch a proof of the elementary facts stated in Lemmas 7 and 8 below. We denote $\nu(\xi)$ the unit outward normal of Ω at the point $\xi \in \partial\Omega$. Let

$$\omega := \{x \in \mathbb{R}^N : (\nu(x_0), x) < d_0\}.$$

LEMMA 7. *If $x \in \omega$ then $\lambda_j x + x_j \in \Omega$ for large j.*

Proof. Denote $x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R}$ for any x. We may assume that, near x_0, Ω (resp. $\partial\Omega$) consists of the points x such that $x_N < \varphi(x')$ (resp. $x_N = \varphi(x')$) where φ is a real C^1 function defined in a neighborhood of x'_0. Moreover, with the above notations, the unit outward normal ν is given by $\nu(\xi) = \nabla\theta(\xi)/|\nabla\theta(\xi)|$ where $\theta(x) := x_N - \varphi(x')$, and $z_j - x_j = d_j \nu(z_j)$ for large j.

Let $\varepsilon > 0$ be given. Using the uniform continuity of $\nabla\theta$ we see that, for large j,

$$\theta(\lambda_j x + x_j) \leqslant \theta(z_j) + \varepsilon(d_j + \lambda_j\, |x|) + (\nabla\theta(z_j), \lambda_j x + x_j - z_j).$$

Divide the above expression by $\lambda_j\, |\nabla\theta(z_j)|$. Since $\theta(z_0) = 0$ and $|\nabla\theta(z_j)| \geqslant 1$, we must prove that

$$\varepsilon\left(\frac{d_j}{\lambda_j} + |x|\right) + (\nu(z_j), x) < \frac{d_j}{\lambda_j}. \tag{11}$$

Now, since d_j/λ_j is bounded and $(\nu(x_0), x) < d_0$, (11) holds for small ε and large j and this proves the lemma. ∎

Thanks to Lemma 7, the arguments in [10] imply that (v_j) has a limit function v (uniformly on compact subsets of ω) which satisfies $v(0) = 1$ and

$$\Delta v + a(x_0)\, |v|^{p-2}\, v = 0, \qquad x \in \omega. \tag{12}$$

Our next lemma shows that we can continuously extend v by setting

$$v(x) = 0, \qquad x \in \partial\omega. \tag{13}$$

LEMMA 8. *Let $R>0$. There exists $C>0$ such that for any $x\in\omega\cap B_R(0)$ there exists $j(x)$ such that*

$$|v_j(x)|\leqslant C(d_0-(\nu(x_0),x))+o(1),\qquad \forall j\geqslant j(x).$$

Proof. Since (v_j) is uniformly bounded, we have by elliptic regularity up to the boundary that $(|\nabla v_j|)$ is uniformly bounded on compact sets. Denote $y_j\in\partial\Omega$ the projection of $\lambda_j x+x_j$ in $\partial\Omega$, i.e.,

$$y_j=\lambda_j x+x_j+\beta_j\nu(y_j),\qquad \beta_j:=\mathrm{dist}(\lambda_j x+x_j,\partial\Omega).$$

The sequences $(z_j-x_j)/\lambda_j$, $(y_j-x_j)/\lambda_j$, $(y_j-z_j)/\lambda_j$ are bounded. Moreover, by a uniform continuity argument similar to the one in Lemma 7,

$$\left(\nu(x_0),\frac{z_j-y_j}{\lambda_j}\right)\to 0\qquad\text{as}\quad j\to\infty. \tag{14}$$

Therefore, as long as x remains bounded,

$$\begin{aligned}|v_j(x)|&=\left|v_j(x)-v_j\left(\frac{y_j-x_j}{\lambda_j}\right)\right|\\&\leqslant\frac{C}{\lambda_j}|\lambda_j x+x_j-y_j|=\frac{C}{\lambda_j}\beta_j\\&=C\left(\frac{d_j}{\lambda_j}-(\nu(y_j),x)\right)+\frac{C}{\lambda_j}[(\nu(z_j),x_j-z_j)+(\nu(y_j),y_j-x_j)].\end{aligned}$$

Since $z_j\to x_0$ and $y_j\to x_0$, the conclusion follows from (14). ∎

Let v satisfy (12), (13). Observe that since $v(0)\neq 0$ we must have $d_0>0$. By elliptic regularity up to the boundary, $v\in C^1(\bar\omega)$. As in Lemma 6, there exists $R_0>0$ such that $E''(v)$ defined as in (10) satisfies $E''(v)\,\varphi,\ \varphi\geqslant 0$ for every $\varphi\in H_0^1(\omega\backslash B_{R_0}(0))$. Hence Proposition 11 implies $v=0$. This contradicts $v(0)=1$ and ends the proof of Theorem 1 if Case B holds.

Case C. Suppose $x_0\in\Omega$ and $a(x_0)=0$. Let λ_j be given by

$$\lambda_j^3 M_j^{p-2}=1.$$

Following [7], introduce

$$\delta_j:=\mathrm{dist}(x_j,\Omega^0)=|x_j-z_j|\to 0$$

for some $z_j\in\Omega^0$.

LEMMA 9. *If $a(x_j)\leqslant 0$ then $\lim_{j\to\infty}\delta_j/\lambda_j=0$.*

Proof. If $a(x_j) \leqslant 0$ then assumption (H2) implies $a^-(x_j) \geqslant \varepsilon\delta_j$ for some $\varepsilon > 0$ and every j large. As in Lemma 5, this implies $\delta_j \leqslant C/M_j^{q-2}$ for some $C > 0$. Since $3q - p > 4$ (cf. (1)), the conclusion follows. ∎

Now, for given $\alpha_j > 0$, define

$$v_j(x) := \frac{u_j(\lambda_j \alpha_j x + x_j)}{M_j}. \tag{15}$$

Then v_j satisfies

$$\Delta v_j + \mu \lambda_j^2 \alpha_j^2 v_j + \theta_j(x) = 0,$$

where

$$\theta_j(x) := \frac{\alpha_j^2}{\lambda_j} \left[a^+(\lambda_j \alpha_j x + x_j) \frac{g(M_j v_j)}{M_j^{p-1}} - a^-(\lambda_j \alpha_j x + x_j) \frac{g_j(M_j v_j)}{M_j^{p-1}} \right].$$

For large j, $\delta_j := \pm(\nabla a(z_j), x_j - z_j)/|\nabla a(z_j)|$ where the plus and the minus sign occur according as $x_j \in \Omega^+$ or $x_j \in \Omega^-$. Taylor formula then reads as

$$a(\lambda_j \alpha_j x + x_j) = \pm |\nabla a(z_j)| \, \delta_j + \lambda_j \alpha_j (\nabla a(z_j), x) + O(\lambda_j^2 \alpha_j^2 \, |x|^2 + \delta_j^2). \tag{16}$$

Suppose first that $\delta_j/\lambda_j \to \infty$. Then, by Lemma 9, $x_j \in \Omega^+$. By choosing $\alpha_j^2 = \lambda_j/\delta_j$ we see from (16) that $(\alpha_j^2/\lambda_j) \, a(\lambda_j \alpha_j x + x_j)$ is uniformly bounded and positive as long as x remains bounded and j is large. Proceeding as in Case A, we duce that (v_j) has a limit function v, uniformly on compact sets, which satisfies

$$\Delta v + |\nabla a(x_0)| \, |v|^{p-\times 2} \, v = 0, \qquad x \in \mathbb{R}^N.$$

Using Proposition 10 (and a slight change in Lemma 6) we deduce that $v = 0$ and this contradicts $v(0) = 1$.

So, from now on let us assume that, up to a subsequence,

$$\frac{\delta_j}{\lambda_j} \to \delta_0 \in [0, \infty[.$$

Then we take $\alpha_j = 1$ in (15). Since (θ_j) is uniformly bounded on compact sets, (v_j) has a sublimit v in every ball of $\mathbb{R}^N$. Up to a subsequence, define (see Section 1),

$$\ell := \lim \frac{a_j}{M_j} \in [0, 1].$$

Let $f \in C^1([-1,1];\mathbb{R})$ be given by

$$f(s) := \begin{cases} \dfrac{p-1}{q-1}\,\ell^{p-q}\,|s|^{q-2}\,s + \dfrac{q-p}{q-1}\,\ell^{p-1}, & \text{for} \quad s \geqslant \ell; \\ |s|^{p-2}\,s, & \text{for} \quad |s| \leqslant \ell; \\ \dfrac{p-1}{q-1}\,\ell^{p-q}\,|s|^{q-2}\,s - \dfrac{q-p}{q-1}\,\ell^{p-1}, & \text{for} \quad s \leqslant -\ell. \end{cases}$$

Using the asymptotic formulas (51) one easily sees that, for every $x \in \mathbb{R}^N$,

$$\lim_{j\to\infty} \frac{g_j(M_j v_j(x))}{M_j^{p-1}} = f(v(x)). \tag{17}$$

Recall also that if $\delta_0 > 0$ then, by Lemma 9, $x_j \in \Omega^+$. Thus, denoting

$$\beta(x) := \delta_0\,|\nabla a(x_0)| + (\nabla a(x_0), x), \tag{18}$$

we conclude by (16), (17) that, uniformly on compact sets, (v_j) has a non-zero limit function v satisfying

$$\Delta v + \beta^+(x)\,|v|^{p-2}\,v - \beta^-(x)\,f(v) = 0, \qquad x \in \mathbb{R}^N. \tag{19}$$

Using an affine change of coordinates, we may assume β is the linear projection $\beta(x) = x_N$ for $x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R}$, so that (19) becomes

$$\Delta v + x_N^+\,|v|^{p-2}\,v - x_N^-\,f(v) = 0, \qquad x \in \mathbb{R}^N. \tag{20}$$

Recall also (cf. Lemma 6) that there exists $R_0 > 0$ such that $E''(v)\,\varphi, \varphi \geqslant 0$, $\forall \varphi \in H_0^1(\mathbb{R}^N \backslash B_{R_0}(0))$, where now $E''(v)$ is given by

$$E''(v)\,\varphi, \varphi := \int_{\mathbb{R}^N} |\nabla\varphi|^2 - (p-1)\int_{\mathbb{R}^N} x_N^+\,|v|^{p-2}\,\varphi^2 + \int_{\mathbb{R}^N} x_N^-\,f'(v)\,\varphi^2. \tag{21}$$

At this point we state some properties of the function f. The proofs of (B1)–(B3) below are similar and easier than the ones in Section 5 and therefore we omit them. We denote $F(s) := \int_0^s f(\xi)\,d\xi$. Observe that if $\ell = 0$ then $f = 0$. On the other hand, *if* $\ell \neq 0$ *then there is* $C > 0$ (e.g., $C = \ell^{p-1}/3$) *such that, for every* $s \in [-1, 1]$,

(B1) $qF(s) \leqslant f(s)\,s \leqslant pF(s)$,

(B2) $C\,|s|^p \leqslant f(s)\,s \leqslant |s|^p$,

(B3) $f'(s)\,s^2 - (p-1)\,f(s)\,s \leqslant 0$.

Using Propositions 12 or 16 (see Section 4) according to whether $\ell \neq 0$ or $\ell = 0$, we conclude that $v = 0$. However, by construction, $v \neq 0$. This ends the proof of Theorem 1 if Case C holds.

Case D. Suppose $x_0 \in \partial\Omega$ and $a(x_0) = 0$. We briefly show how to adapt the arguments of Case C. We use the previous notations $\delta_j = \operatorname{dist}(x_j, \Omega^0)$, $d_j = \operatorname{dist}(x_j, \partial\Omega)$ and (up to subsequences),

$$\lambda_j^3 M_j^{p-2} = 1, \qquad \delta_0 = \lim \frac{\delta_j}{\lambda_j}, \qquad d_0 = \lim \frac{d_j}{\lambda_j}.$$

Suppose first $\delta_0 = \infty$. Using the blow-up functions (15) with $\alpha_j^2 = \lambda_j/\delta_j$, we arrive at the limit problem

$$\Delta v + |\nabla a(x_0)|\, |v|^{p-2}\, v = 0, \qquad \text{for all } x \text{ s.t. } (v(x_0), x) < \bar{d}_0,$$

where (up to a subsequence) $\bar{d}_0 := \lim d_j/\lambda_j \alpha_j \in [0, \infty]$. Moreover, in case $\bar{d}_0 < \infty$, $v(x) = 0$ for x such that $(v(x_0), x) = \bar{d}_0$. The argument in Case B leads to $v = 0$.

Suppose now $\delta_0 < \infty$. Using the blow-up functions (15) with $\alpha_j = 1$, we arrive at the limit equation

$$\Delta v + \beta^+(x)\, |v|^{p-2}\, v - \beta^-(x)\, f(v) = 0, \qquad x \in \omega := \{x : (v(x_0), x) < d_0\},$$

where β is given in (18). If $d_0 = \infty$, Propositions 12 and 16 imply $v = 0$. So assume $d_0 < \infty$. Then $v = 0$ on $\partial\omega$. Using an affine change of coordinates, we may assume v satisfies

$$\Delta v + x_N^+\, |v|^{p-2}\, v - x_N^-\, f(v) = 0, \qquad \text{for all } x \text{ s.t. } (y, x) < c,$$

for some $c \in \mathbb{R}$ and $y \in \mathbb{R}^N$ with $y \neq 0$ and $y_N \leqslant 0$. Moreover, $v(x) = 0$ for x such that $(y, x) = c$. Propositions 15 and 17 imply $v = 0$. However, by construction $v \neq 0$. This ends the proof if Case D holds and completes the proof of Theorem 1.

4. SOME NONLINEAR LIOUVILLE THEOREMS

In the proof of Theorem 1 (Section 3) we used some results that we now state and prove. They concern bounded solutions $v \in C^2(\omega; \mathbb{R})$ of equation

$$\Delta v + f(x, v) = 0, \qquad x \in \omega,$$

where $\omega \subset \mathbb{R}^N$ is an open set and $f \in C(\omega \times \mathbb{R}; \mathbb{R})$ is C^1 in v. For every function $\varphi \in H^1(\omega)$, let

$$E''(v)\,\varphi, \varphi := \int_\omega |\nabla \varphi|^2 - \int_\omega \frac{\partial f}{\partial v}(x, v)\,\varphi^2.$$

DEFINITION. We say that v has *finite index* if there exists $R_0 > 0$ such that

$$E''(v)\,\varphi, \varphi \geqslant 0, \qquad \forall \varphi \in H_0^1(\omega \backslash B_{R_0}(0)).$$

We first state a result from [6]. We include a short proof since ours is somewhat less tricky then the one in [6]. Our argument can then be easily adapted to treat the more involved results at the end of the section. We denote $2^* = 2N/(N-2)$ for any $N \geqslant 2$.

PROPOSITION 10 [6]. *Let $v \in C^2(\mathbb{R}^N)$ be bounded and satisfy, for some $\ell > 0$, $2 < p < 2^*$,*

$$\Delta v + \ell\, |v|^{p-2}\, v = 0, \qquad x \in \mathbb{R}^N.$$

If v has finite index then $v = 0$.

Proof. We may assume $\ell = 1$. For each R, denote $B_R := B_R(0)$, $B_R^c := \mathbb{R}^N \backslash B_R(0)$.

(1) For large R, let $\varphi \in \mathscr{D}(\mathbb{R}^N)$ be such that $\varphi = 1$ over $B_R \backslash B_{2R_0}$, $\varphi = 0$ over $B_{R_0} \cup B_{2R}^c$. Observe that $|\nabla \varphi(x)| \leqslant C/R$ for every $x \in B_R^c$, for some constant C depending only on the dimension N. From now on we denote by C some positive, possibly different from place to place, constant which does not depend on R. We prove that $v \in L^p(\mathbb{R}^N)$.

Multiply the equation by $v\varphi^2$ to obtain (all integrals are taken over $\mathbb{R}^N$ except when mentioned)

$$\int |v|^p\, \varphi^2 = \int \varphi^2\, |\nabla v|^2 + 2 \int \varphi v(\nabla v\, \nabla \varphi). \tag{22}$$

By assumption, $E''(v)\, v\varphi, v\varphi \geqslant 0$, i.e.,

$$\int v^2\, |\nabla \varphi|^2 + \int \varphi^2\, |\nabla v|^2 + 2 \int \varphi v(\nabla v\, \nabla \varphi) \geqslant (p-1) \int |v|^p\, \varphi^2.$$

Since $p > 2$, we thus see that

$$\int |v|^p\, \varphi^2 + \int |\nabla v|^2\, \varphi^2 \leqslant C \int v^2\, |\nabla \varphi|^2.$$

In particular,

$$\int_{B_R \setminus B_{2R_0}} |v|^p \leqslant C\left(1 + \int_{B_{2R} \setminus B_R} v^2 \, |\nabla \varphi|^2\right),$$

so that, for every $R > 2R_0$,

$$\int_{B_R} |v|^p \leqslant C\left(1 + R^{-2} \int_{B_{2R}} v^2\right). \tag{23}$$

Since v is bounded, $\int_{B_R} |v|^p = O(R^N)$. Thus, if $N = 2$ the inequality in (23) shows that $\int |v|^p$ is finite. Let now $N > 2$ and suppose by contradiction that $\int |v|^p$ is not finite. Then, for every large R,

$$\int_{B_R} |v|^p \leqslant C R^{-2} \int_{B_{2R}} v^2. \tag{24}$$

Hölder inequality implies

$$R^{-2} \int_{B_{2R}} v^2 \leqslant C\left(\int_{B_{2R}} |v|^p\right)^{2/p} R^{-\alpha}, \tag{25}$$

where $\alpha = 2 - N(1 - (2/p)) = 2N((1/p) - (1/2^*)) > 0$. Plugging this into (24) we get, for $k \in \mathbb{N}$ and some constant C (depending on k),

$$\int_{B_R} |v|^p \leqslant C R^{-\alpha} \left(\int_{B_{2^k R}} |v|^p\right)^{(2/p)^k} \leqslant C R^{N(2/p)^k - \alpha}.$$

Thus, taking k large enough, we may assume that the power is negative. Letting $R \to \infty$ shows $v = 0$, a contradiction. Thus $\int |v|^p$ is finite, as claimed.

(2) For any $R > 0$ choose now $\varphi = 1$ over B_R, $\varphi = 0$ over B_{2R}^c, $\|\nabla \varphi\|_\infty \leqslant C R^{-1}$. It follows from (22) that

$$\int_{B_R} |\nabla v|^2 \leqslant C\left(\int_{B_{2R}} |v|^p + R^{-2} \int_{B_{2R}} v^2\right).$$

Thus (25) implies that $\int |\nabla v|^2$ is finite. Moreover, observing that, by Cauchy–Schwartz inequality,

$$\left| \int v \varphi (\nabla v, \nabla \varphi) \right| \leqslant C\left(R^{-2} \int_{B_{2R}} v^2\right)^{1/2} \left(\int |\nabla v|^2\right)^{1/2} = o(1),$$

we can pass (22) to the limit and conclude

$$\int |\nabla v|^2 = \int |v|^p. \tag{26}$$

(3) Since both $\int |\nabla v|^2$ and $\int |v|^p$ are finite, we may write Pohožaev identity (see, e.g., [12, 19]),

$$\frac{2^*}{p} \int |v|^p = \int |\nabla v|^2.$$

This and (26) imply $v = 0$. ∎

The above proof extends immediately to the following situation. Given some nonzero vector $y \in \mathbb{R}^N$ and some $c \in \mathbb{R}$, let

$$\omega := \{x \in \mathbb{R}^N : (y, x) < c\}.$$

PROPOSITION 11 [6]. *Let $v \in C^2(\omega) \cap C^1(\bar{\omega})$ be bounded and satisfy, for some $\ell > 0$, $2 < p < 2^*$,*

$$\Delta v + \ell\, |v|^{p-2}\, v = 0 \qquad \text{in } \omega, \qquad v = 0 \qquad \text{on } \partial\omega.$$

If v has finite index then $v = 0$.

Proof. (1) Let $\varphi \in \mathscr{D}(\mathbb{R}^N \backslash B_{R_0}(0))$ and denote $\Gamma := \omega \backslash B_{R_0(0)}$. Since $v\varphi \in H^1(\Gamma) \cap C(\bar{\Gamma})$ and $v\varphi$ vanishes on the locally Lipschitz boundary $\partial\Gamma$, it follows that $v\varphi \in H_0^1(\Gamma)$ (see, e.g., [11]). Thus, by our assumption, $E''(v)\, v\varphi, v\varphi \geqslant 0$. As a consequence, the arguments in the proof of Proposition 10 can be repeated step by step and yield

$$\int_\omega |\nabla v|^2 = \int_\omega |v|^p < \infty. \tag{27}$$

(2) We now use Pohožaev identity,

$$\frac{2^*}{p} \int_\omega |v|^p = \int_\omega |\nabla v|^2 + \frac{1}{N-2} \int_{\partial\omega} |\nabla v|^2\, ((\sigma - y_0), \nu(x_0))\, d\sigma,$$

where $y_0 \in \mathbb{R}^N$ is arbitrary (see [12, Proposition 2.1]). By choosing, e.g., $y_0 = d_0\, \nu(x_0)$, we deduce

$$\frac{2^*}{p} \int_\omega |v|^p \leqslant \int_\omega |\nabla v|^2.$$

This together with (27) implies $v = 0$. ∎

We now consider the case where $f(x, v) = x_N |v|^{p-2} v$. In fact, for our purposes, we need to study a more general nonlinear term. So, let $f \in C^1([-1, 1]; \mathbb{R})$ satisfy, for some positive constants c_1, c_2, c_3, c_4 and every $s \in [-1, 1]$,

(B1) $c_1 F(s) \leqslant f(s)\, s \leqslant c_2 F(s)$,

(B2) $c_3 |s|^p \leqslant f(s)\, s \leqslant c_4 |s|^p$,

(B3) $f'(s)\, s^2 - (p-1)\, f(s)\, s \leqslant 0$,

where $F(s) := \int_0^s f(\xi)\, d\xi$.

Proposition 12. *Let $v \in C^2(\mathbb{R}^N)$ be bounded with $\|v\|_\infty \leqslant 1$ and satisfy*

$$\Delta v + x_N^+ |v|^{p-2} v - x_N^- f(v) = 0, \qquad x \in \mathbb{R}^N, \tag{28}$$

where $2 < p \leqslant 2^*$ and $f \in C^1([-1, 1]; \mathbb{R})$ satisfies (B1)–(B3). *If v has finite index then $v = 0$.*

The proof of Proposition 12 relies on the following estimates. We denote $g(s) := |s|^{p-2} s$ and $G(s) := |s|^p/p$. Unless otherwise stated, all integrals are taken over the whole space $\mathbb{R}^N$.

Lemma 13. *Let $v \in C^2(\mathbb{R}^N)$ satisfy* (28) *with f satisfying property* (B1). *There exists $C > 0$ such that, for every $\varphi \in \mathcal{D}(\mathbb{R}^N)$, $\varphi \geqslant 0$,*

(C1) $\int (x_N^+)^2\, G(v)\, \varphi^2 + \int (x_N^-)^2\, F(v)\, \varphi^2 \leqslant \int |x_N|\, \varphi^2\, |\nabla v|^2 + \int |v|\, \varphi^2\, |\nabla v| + 2 \int |x_N|\, |v|\, \varphi\, |\nabla v|\, |\nabla \varphi|$.

(C2) $\int x_N^+\, G(v)\, \varphi + \int x_N^-\, F(v)\, \varphi \leqslant C(\int |x_N|\, |\nabla v|^2\, |\nabla \varphi| + \int \varphi\, |\nabla v|^2 + \int (x_N^+)^2\, G(v)\, |\nabla \varphi| + \int (x_N^-)^2\, F(v)\, |\nabla \varphi|)$.

(C3) $\int_{\{x_N \geqslant 0\}} G(v)\, \varphi + \int_{\{x_N \leqslant 0\}} F(v)\, \varphi \leqslant C(\int |\nabla v|^2\, |\nabla \varphi| + \int x_N^+\, G(v)\, |\nabla \varphi| + \int x_N^-\, F(v)\, |\nabla \varphi|)$.

Proof. Multiply the equation by $x_N v \varphi^2$ and integrate over $\mathbb{R}^N$. Taking property (B1) into account, the first inequality follows readily.

Denote e_N the unit vector $(0, 0, ..., 1)$. Since

$$\begin{aligned} \operatorname{div} U &:= \operatorname{div}\left(\varphi \frac{\partial v}{\partial x_N} \nabla v - \varphi \frac{|\nabla v|^2}{2} e_N\right) \\ &= \varphi \frac{\partial v}{\partial x_N} \Delta v + \frac{\partial v}{\partial x_N} (\nabla \varphi, \nabla v) - \frac{\partial \varphi}{\partial x_N} \frac{|\nabla v|^2}{2}, \end{aligned}$$

we deduce

$$\int \varphi \frac{\partial v}{\partial x_N} \Delta v = \int \frac{\partial \varphi}{\partial x_N} \frac{|\nabla v|^2}{2} - \int \frac{\partial v}{\partial x_N} (\nabla \varphi, \nabla v). \tag{29}$$

Since

$$\operatorname{div} V := \operatorname{div}(x_N G(v) \varphi e_N) = G(v) \varphi + x_N g(v) \frac{\partial v}{\partial x_N} \varphi + x_N G(v) \frac{\partial \varphi}{\partial x_N}$$

and V vanishes on $\{x_N = 0\}$, we deduce

$$\int x_N^+ g(v) \frac{\partial v}{\partial x_N} \varphi = -\int_{\{x_N \geqslant 0\}} G(v) \varphi - \int_{\{x_N \geqslant 0\}} x_N G(v) \frac{\partial \varphi}{\partial x_N}. \tag{30}$$

Similarly,

$$-\int x_N^- f(v) \frac{\partial v}{\partial x_N} \varphi = -\int_{\{x_N \leqslant 0\}} F(v) \varphi - \int_{\{x_N \leqslant 0\}} x_N F(v) \frac{\partial \varphi}{\partial x_N}. \tag{31}$$

Multiply the equation by $\varphi(\partial v/\partial x_N)$. If we add term by term identities (29), (30) and (31), inequality (C3) follows readily. As for (C2), multiply the equation by $\varphi x_N(\partial v/\partial x_N)$ and integrate both $\operatorname{div}(x_N U)$ and $\operatorname{div}(x_N V)$ over $\{x_N \geqslant 0\}$ (see the proof of Proposition 4 for the computations). Proceed similarly over $\{x_N \leqslant 0\}$ and add the obtained identities. Then we get (C2). ∎

LEMMA 14. *Let $v \in C^2(\mathbb{R}^N)$ satisfy* (26) *with f satisfying properties* (B1) *and* (B2). *There exists some constant $C > 0$ such that, for every $R > 0$,*

(D1) $\int_{B_R} x_N^2 |v|^p \leqslant C \int_{B_{2R}} |x_N| \, |\nabla v|^2 + C(\int_{B_{2R}} v^2)^{1/2} (\int_{B_{2R}} |\nabla v|^2)^{1/2}$.

(D2) $\int_{B_R} |x_N| \, |v|^p \leqslant CR^{-1} \int_{B_{2R}} x_N^2 |v|^p + C \int_{B_{2R}} |\nabla v|^2$.

(D3) $\int_{B_R} |v|^p \leqslant CR^{-1} (\int_{B_{2R}} |x_N| \, |v|^p + \int_{B_{2R}} |\nabla v|^2)$.

Proof. This follows readily from Lemma 13 together with properties (B1)–(B2), by choosing functions φ with $0 \leqslant \varphi \leqslant 1$, $\varphi = 1$ over B_R, $\varphi = 0$ over $\mathbb{R}^N \backslash B_{2R}$. ∎

Proof of Proposition 12. (1) As before, denote $B_R = B_R(0)$, $B_R^c = \mathbb{R}^N \backslash B_R(0)$. For large R, let $\varphi \in \mathscr{D}(\mathbb{R}^N)$ be such that $\varphi = 1$ over $B_R \backslash B_{2R_0}$, $\varphi = 0$ over $B_{R_0} \cup B_{2R}^c$. From now on we denote by C some positive, possibly different from place to place, constant which does not depend on R.

Multiply the equation by $v\varphi^2$ to obtain

$$\int (x_N^+ g(v) v - x_N^- f(v) v) \varphi^2 = \int \varphi^2 |\nabla v|^2 + 2 \int \varphi v(\nabla v, \nabla \varphi). \tag{32}$$

Property (B3) and the assumption $E''(v)\,v\varphi,\ v\varphi \geqslant 0$ imply

$$\int v^2\,|\nabla\varphi|^2+\int \varphi^2\,|\nabla v|^2+2\int \varphi v(\nabla v, \nabla\varphi)$$

$$\geqslant \int x_N^+\,(g'(v)\,v^2-x_N^-\,f'(v)\,v^2)\,\varphi^2$$

$$\geqslant (p-1)\int (x_N^+\,g(v)\,v-x_N^-\,f(v)\,v)\,\varphi^2.$$

This together with (32) implies

$$\int |\nabla v|^2\,\varphi^2 \leqslant C\int v^2\,|\nabla\varphi|^2.$$

As in (23) it follows that, for large R,

$$\int_{B_{R/2}} |\nabla v|^2 \leqslant C\left(1+R^{-2}\int_{B_R} v^2\right). \tag{33}$$

(2) Assume first $N \geqslant 3$. Then $\int_{B_R} v^2 \leqslant CR^N$ with $N>2$. More generally, suppose there exist $C>0$, $\alpha>2$ such that, for every large R,

$$\int_{B_R} v^2 \leqslant CR^{\alpha}. \tag{34}$$

Then, from (33), $\int_{B_{R/2}} |\nabla v|^2 \leqslant CR^{\alpha-2}$. In a recurrent way, inequalities (D1)–(D3) imply

$$\int_{B_{R/4}} x_N^2\,|v|^p \leqslant CR^{\alpha-1}, \qquad \int_{B_{R/8}} |x_N|\,|v|^p \leqslant CR^{\alpha-2}, \qquad \int_{B_{R/16}} |v|^p \leqslant CR^{\alpha-3}.$$

By Hölder inequality,

$$\int_{B_{R/16}} |v|^2 \leqslant C\left(\int_{B_{R/16}} |v|^p\right)^{2/p} R^{N(1-2/p)}$$

$$\leqslant CR^{(\alpha-3)\,2/p}R^{2+2N(1/2^*-1/p)}.$$

Thus we see that (34) implies, for every large R,

$$\int_{B_R} |v|^2 \leqslant CR^{(\alpha-3)(2/p)+2}. \tag{35}$$

If we iterate the argument starting from (35) we conclude that after a finite number k of steps there exist C_0, R_0 depending on k such that, for every $R \geqslant R_0$,

$$\begin{aligned}\int_{B_R} |v|^2 &\leqslant C_0 R^{(\alpha-3)(2/p)^k-(2/p)^{k-1}-\cdots-2/p+2}\\ &\leqslant C_0 R^{[(\alpha-3)(2/p)^{k-1}-1]\,2/p+2}.\end{aligned}$$

Taking $\alpha = N$ in (34) and choosing k large so that $(N-3)(2/p)^{k-1} < 1$ we conclude that, for every large R,

$$\int_{B_R} |v|^2 \leqslant CR^2. \tag{36}$$

Since v is bounded, (36) also holds if $N = 2$.

(3) Combining (33) and (36) we see that $\int |\nabla v|^2$ is finite. In a recurrent way, inequalities (D1) and (D2) imply $\int |x_N|\,|v|^p$ is also finite. Then (D3) shows that $v = 0$. ∎

Proposition 12 extends easily to the following situation. Given some nonzero vector $y = (y_1, ..., y_N) \in \mathbb{R}^N$ and some $c \in \mathbb{R}$, let

$$\omega := \{x \in \mathbb{R}^N : (y, x) < c\}.$$

PROPOSITION 15. *Let $v \in C^2(\omega) \cap C^1(\bar{\omega})$ be bounded with $\|v\|_\infty \leqslant 1$ and satisfy*

$$\Delta v + x_N^+ |v|^{p-2} v - x_N^- f(v) = 0 \quad \text{in } \omega, \qquad v = 0 \quad \text{on } \partial\omega,$$

where $2 < p \leqslant 2^*$ and $f \in C^1([-1, 1]; \mathbb{R})$ satisfies (B1)–(B3). *Suppose moreover*

$$y_N \leqslant 0.$$

If v has finite index then $v = 0$.

Proof. This follows exactly as in Proposition 12 once we show that the estimates in Lemma 13 hold true (with the integrals taken over ω). An inspection of the proof in Lemma 13 shows that (C1) and (C2) remain unchanged. As for (C3), simply observe that now the right hand member of (29) has an extra term

$$\int_{\partial\omega}\left(\varphi\,\frac{\partial v}{\partial x_N}\,(\nabla v,\, y)-\varphi\,\frac{|\nabla v|^2}{2}\,y_N\right).$$

Since v vanishes on $\partial\omega$, we have $\nabla v=(\nabla v,\, y)\, y$, so that, on $\partial\omega$, the above function reads as

$$\tfrac{1}{2}\varphi\, |\nabla v|^2\, y_N\leqslant 0.$$

Thus (29) becomes an inequality, and this suffices to deduce (C3). ∎

We now consider the case where $f(x, v)=x_N^+\, |v|^{p-2}\, v$. If $N=2$, this case follows exactly as in the previous one. However, if $N\geqslant 3$, we need to use a somewhat more involved argument.

PROPOSITION 16. *Let $v\in C^2(\mathbb{R}^N)$ be bounded and satisfy*

$$\Delta v+x_N^+\, |v|^{p-2}\, v=0, \qquad x\in\mathbb{R}^N,$$

where $2<p<2^*$. If v has finite index then $v=0$.

Proof. (1) As observed before, we may already assume $N>2$. In the following we denote $B_R^+:=B_R(0)\cap\{x\in\mathbb{R}^N : x_N\geqslant 0\}$. We first collect the four main estimates which will be needed in the sequel: *There exists $C>0$ and $k\in[2/3, 1[$ such that, for every large R,*

$$\begin{aligned}
&\text{(i)} && \textstyle\int_{B_R} x_N^+\, |v|^p+\int_{B_R} |\nabla v|^2\leqslant C(1+R^{-2}\int_{B_{2R}} v^2),\\
&\text{(ii)} && \textstyle\int_{B_R^+} |v|^p\leqslant CR^{-1}(\int_{B_{2R}} x_N^+\, |v|^p+\int_{B_{2R}} |\nabla v|^2),\\
&\text{(iii)} && \textstyle\int_{B_R} x_N^+\, |v|^{p-2}\leqslant CR^{N-2},\\
&\text{(iv)} && \textstyle\int_{B_R} x_N^+\, |v|^p\leqslant CR^{N(p-2)/3p}(\int_{B_{2R}^+} |v|^p)^k.
\end{aligned}$$

Indeed, (ii) follows as in (D3) of Lemma 14, while (i) follows by the assumption $E''(v)\, v\varphi,\ v\varphi\geqslant 0$, as in the proof of Propositions 10 and 12. Here, as before, $\varphi\in\mathcal{D}(\mathbb{R}^N)$ is such that $\varphi=1$ over $B_R\backslash B_{2R_0}$, $\varphi=0$ over $B_{R_0}\cup B_{2R}^c$. Using now the fact that $E''(v)\,\varphi,\ \varphi\geqslant 0$ for every such φ, (iii) follows readily. As for (iv), again we use our assumption $E''(v)\, x_N^+\,\varphi v$, $x_N^+\,\varphi v\geqslant 0$. A tedious but easy computation shows that this implies

$$\int_{B_R} (x_N^+)^3\, |v|^p\leqslant C\left(1+\int_{B_{2R}^+} v^2\right).$$

From this and Hölder inequality we deduce

$$
\begin{aligned}
\int_{B_R} x_N^+ |v|^p &\leqslant \left(\int_{B_R} (x_N^+)^3 |v|^p\right)^{1/3} \left(\int_{B_R^+} |v|^p\right)^{2/3} \\
&\leqslant C\left(\int_{B_{2R}^+} |v|^p\right)^{2/3} + C\left(\int_{B_{2R}^+} v^2\right)^{1/3} \left(\int_{B_{2R}^+} |v|^p\right)^{2/3} \\
&\leqslant C\left(\int_{B_{2R}^+} |v|^p\right)^{2/3} + CR^{N(p-2)/3p} \left(\int_{B_{2R}^+} |v|^p\right)^{2/3(1+1/p)} \\
&\leqslant CR^{N(p-2)/3p} \left(\int_{B_{2R}^+} |v|^p\right)^{k}
\end{aligned}
$$

for every large R, where $k = 2/3(1+1/p)$. Observe that indeed $k \in [2/3, 1[$.

(2) In the following we argue by contradiction and suppose $v \neq 0$. We claim that, for every large R,

$$
R^{-2} \int_{B_{2R}} v^2 \geqslant 1. \tag{37}
$$

Indeed, otherwise we would have $\int_{B_{2R_j}} v^2 \leqslant R_j^2$ for some sequence $R_j \to \infty$. Using estimates (i) and (ii) we would deduce

$$
\int_{B_{R_j/2}^+} |v|^p \leqslant CR_j^{-1}.
$$

Thus $v = 0$ over $\{x_N \geqslant 0\}$. By unique continuation, $v = 0$, contradicting our assumption $v \neq 0$. Thus (37) holds. In particular, (37) and Hölder inequality,

$$
R^{-2} \int_{B_{2R}} v^2 \leqslant \left(\int_{B_{2R}} |v|^p\right)^{2/p} R^{N(1-2/p)-2}, \tag{38}
$$

imply

$$
\int_{\mathbb{R}^N} |v|^p = \infty. \tag{39}
$$

(3) We now claim that there exist $C > 0$ and $\alpha_j \to \infty$ such that, for every j,

$$
\int_{B_{8\alpha_j}} |v|^p \leqslant C \int_{B_{\alpha_j}} |v|^p. \tag{40}
$$

Indeed, denote $\varepsilon = (1/8)^{N+1}$ and suppose by contradiction that, for every large R,

$$\int_{B_R} |v|^p \leqslant \varepsilon \int_{B_{8R}} |v|^p.$$

Then, for every large R and every $k \in \mathbb{N}$,

$$\int_{B_R} |v|^p \leqslant \varepsilon^k \int_{B_{8^k R}} |v|^p \leqslant c\varepsilon^k (8^k R)^N = cR^N(\tfrac{1}{8})^k,$$

where c depends only on the dimension N. It follows that $\int_{B_R} |v|^p = 0$ for every large R. Thus $v = 0$, contrarily to our assumption $v \neq 0$. Thus (40) holds.

(4) For (α_j) given in Step 3, and taking (37), (38), (40) into account, we rewrite estimates (i)–(iv) as follows:

(v) $\int_{4\alpha_j} x_N^+ |v|^p + \int_{4\alpha_j} |\nabla v|^2 \leqslant C(\int_{B_{\alpha_j}} |v|^p)^{2/p} \alpha_j^{N(1-(2/p))-2}$,

(vi) $\int_{B^+_{2\alpha_j}} |v|^p \leqslant C\alpha_j^{-1} (\int_{B_{4\alpha_j}} x_N^+ |v|^p + \int_{B_{4\alpha_j}} |\nabla v|^2)$,

(vii) $\int_{B_{\alpha_j}} x_N^+ |v|^{p-2} \leqslant C\alpha_j^{N-2}$,

(viii) $\int_{B_{\alpha_j}} x_N^+ |v|^p \leqslant C\alpha_j^{N(p-2)/3p} (\int_{B^+_{2\alpha_j}} |v|^p)^k$ $(k \in [2/3, 1[\,)$.

(5) We now define a blow-up sequence as follows. Let λ_j be given by

$$\lambda_j^p := \int_{B_{\alpha_j}} |v|^p. \tag{41}$$

It follows from (39) that $\lambda_j \to \infty$. Let

$$v_j(x) := \beta_j v(\alpha_j x), \qquad \text{where} \quad \beta_j := \alpha_j^{N/p} \lambda_j^{-1}. \tag{42}$$

Then $v_j \in C^2(\mathbb{R}^N)$ satisfies

$$\Delta v_j + \mu_j x_N^+ |v_j|^{p-2} v_j = 0, \qquad x \in \mathbb{R}^N, \tag{43}$$

where

$$\mu_j = \alpha_j^3 \beta_j^{2-p} = \alpha_j^{3-(N/p)(p-2)} \lambda_j^{p-2}. \tag{44}$$

Since $p < 2N/(N-2) < 2N/(N-3)$, it follows that

$$\lim_{j \to \infty} \mu_j = \infty. \tag{45}$$

Denote $\Omega = B_1(0)$ and $\Omega^+ = B_1(0) \cap \{x : x_N > 0\}$. Observe that, by definition,

$$\int_\Omega |v_j|^p = 1. \tag{46}$$

Estimates (v)–(viii) imply that v_j satisfies, for some $C > 0$ independent of j,

(ix) $\mu_j \int_\Omega x_N^+ |v_j|^p + \int_\Omega |\nabla v_j|^2 \leqslant C$,

(x) $\mu_j \int_{\Omega^+} |v_j|^p \leqslant C$,

(xi) $\mu_j \int_\Omega x_N^+ |v_j|^{p-2} \leqslant C$,

(xii) $\mu_j \int_\Omega x_N^+ |v_j|^p \to 0$.

Indeed, it follows from (v) and our definitions (41), (42), (44) that

$$\begin{aligned} \mu_j \int_{B_4} x_N^+ |v_j|^p + \int_{B_4} |\nabla v_j|^2 &= \beta_j^2 \alpha_j^{2-N} \left(\int_{B_{4\alpha_j}} x_N^+ |v|^p + \int_{B_{4\alpha_j}} |\nabla v|^2 \right) \\ &= C \beta_j^2 \alpha_j^{2-N} \lambda_j^2 \alpha_j^{N(1-2/p)-2} = C, \end{aligned} \tag{47}$$

and this implies (ix). Similarly, using (vii) we see that

$$\mu_j \int_\Omega x_N^+ |v_j|^{p-2} = \alpha_j^{2-N} \int_{B_{\alpha_j}} x_N^+ |v|^{p-2} \leqslant C,$$

which is estimate (xi). As for (x), we use (vi) and (47) to derive

$$\begin{aligned} \mu_j \int_{B_2^+} |v_j|^p &= \beta_j^2 \alpha_j^{3-N} \int_{B_{2\alpha_j}^+} |v|^p \\ &\leqslant C \beta_j^2 \alpha_j^{2-N} \left(\int_{B_{4\alpha_j}} x_N^+ |v|^p + \int_{B_{4\alpha_j}} |\nabla v|^2 \right) \\ &= C \left(\mu_j \int_{B_4} x_N^+ |v_j|^p + \int_{B_4} |\nabla v_j|^2 \right) \leqslant C. \end{aligned} \tag{48}$$

Finally, we use (48) and (viii) to deduce

$$\begin{aligned} \int_\Omega x_N^+ |v_j|^p &= \beta_j^p \alpha_j^{-N-1} \int_{B_{\alpha_j}} x_N^+ |v|^p \\ &\leqslant C \alpha_j^{-1} (\beta_j^p \alpha_j^{-N})^{1-k} \left(\int_{B_2^+} |v_j|^p \right)^k \alpha_j^{N(p-2)/3p} \\ &\leqslant C \alpha_j^{-1} (\beta_j^p \alpha_j^{-N})^{1-k} \mu_j^{-k} \alpha_j^{N(p-2)/3p}, \end{aligned}$$

so that, writing $1-k=\delta\in\,]0,1/3]$,

$$\begin{aligned}\mu_j\int_\Omega x_N^+\,|v_j|^p &\leqslant C(\beta_j^p\alpha_j^{-N}\mu_j)^\delta\,\alpha_j^{(N(p-2)/3p)-1}\\ &= C\lambda_j^{-2\delta}\alpha_j^{3\delta(1-(N(p-2)/3p))}\alpha_j^{(N(p-2)/3p)-1}\\ &\leqslant C\lambda_j^{-2\delta},\end{aligned}$$

where we used the fact that $p\leqslant 2N/(N-3)$ and $3\delta\leqslant 1$ in the last inequality. This yields (xii).

(6) It follows from (xi) and (xii) that

$$\lim_{j\to\infty}\mu_j\int_\Omega x_N^+\,|v_j|^{p-1}=0. \tag{49}$$

Indeed, denoting $\varepsilon_j^2:=\mu_j\int_\Omega x_n^+\,|v_j|^p\to 0$, we have

$$\begin{aligned}\mu_j\int_\Omega x_N^+\,|v_j|^{p-1}&=\mu_j\int_{\{|v_j|\leqslant\varepsilon_j\}}x_N^+\,|v_j|^{p-1}+\mu_j\int_{\{|v_j|\geqslant\varepsilon_j\}}x_N^+\,|v_j|^{p-1}\\ &\leqslant\varepsilon_j\mu_j\int_{\{|v_j|\leqslant\varepsilon_j\}}x_N^+\,|v_j|^{p-2}+\varepsilon_j^{-1}\mu_j\int_{\{|v_j|\geqslant\varepsilon_j\}}x_N^+\,|v_j|^p\\ &\leqslant\varepsilon_j\mu_j\int_\Omega x_N^+\,|v_j|^{p-2}+\varepsilon_j^{-1}\mu_j\int_\Omega x_N^+\,|v_j|^p\\ &\leqslant\varepsilon_jC+\varepsilon_j.\end{aligned}$$

(7) We now arrive at a contradiction. It follows from (ix) and (46) that, up to a subsequence, (v_j) converges weakly in $H^1(\Omega)$ and strongly in $L^p(\Omega)$ to some function $w\in H^1(\Omega)$ such that

$$\int_\Omega |w|^p=1. \tag{50}$$

Let φ be any function in $\mathscr{D}(\Omega)$. We deduce from (43) and (49) that

$$\int_\Omega (\nabla w,\nabla\varphi)=0.$$

Thus $w\in H^1(\Omega)$ and $\Delta w=0$. Elliptic regularity implies $w\in C^2(\Omega)$. On the other hand, (x) and (45) imply $w=0$ in Ω^+. By unique continuation, $w=0$ in Ω. This contradicts (50) and ends the proof of the proposition. ∎

As before, Proposition 16 extends easily to the following situation. Given some nonzero vector $y=(y_1, ..., y_N)\in\mathbb{R}^N$ and some $c\in\mathbb{R}$, let

$$\omega := \{x\in\mathbb{R}^N : (y, x)<c\}.$$

PROPOSITION 17. *Let $v\in C^2(\omega)\cap C^1(\bar{\omega})$ be bounded and satisfy*

$$\Delta v + x_N^+ \, |v|^{p-2}\, v=0 \qquad \text{in } \omega, \qquad v=0 \qquad \text{on } \partial\omega,$$

where $2<p<2^*$. Suppose moreover

$$y_N\leqslant 0.$$

If v has finite index then $v=0$.

Proof. As already observed in the proof of Proposition 15, we still have the estimates (i)–(iv) of Proposition 16. By repeating then the arguments of the preceding proof, if $v\neq 0$ we arrive at a limit function $w\in C^2(\Omega)$, $w\neq 0$, such that

$$\Delta w=0 \qquad \text{in } \Omega \qquad \text{and} \qquad \int_{\Omega^+} |w|^p=0.$$

Here $\Omega=B_1(0)\cap\{x\colon (y, x)<0\}$ and $\Omega^+=\{x\in\Omega : x_N>0\}$. The desired contradiction follows, since Ω^+ is nonempty. Indeed, denoting $e_N=(0, 0, ..., 1)$ the unit vector of $\mathbb{R}^N$, we have $e_N\in\Omega^+$ if $y_N<0$ and $e_N-y\in\Omega^+$ if $y_N=0$. ∎

5. APPENDIX

In this section we prove properties (A1)–(A4) stated in Section 1. For given $\varepsilon>0$, it follows easily from (H5) that there exists $R>0$ such that, for $|u|\geqslant R$,

$$\left(1-\frac{q}{p}-\varepsilon\right) g(u)\, u\leqslant g(u)\, u-qG(u)\leqslant\left(1-\frac{q}{p}+\varepsilon\right) g(u)\, u.$$

We prove now the above second inequality for g_j. We may assume $R<a_j$ and $u>a_j$ (the case $u<-a_j$ is similar). Since $g'(a_j)=(q-1)\, A_j a_j^{q-2}$ and $g(a_j)=A_j a_j^{q-1}+B_j$, (H5) implies

$$A_j=\frac{p-1}{q-1}\, a_j^{p-q}+a_j^{p-q}o(1), \qquad B_j=\frac{q-p}{q-1}\, a_j^{p-1}+a_j^{p-1}o(1). \tag{43}$$

Now, $G_j(u) = G(a_j) + A_j(u^q/q) + B_j u - A_j(a_j^q/q) - B_j a_j$ and $qG(a_j) \geqslant ((q/p) - (\varepsilon/2))\, a_j^p$ for large j, so that it is enough to prove

$$\left(\frac{q}{p} - 1 - \varepsilon\right) A_j u^q + \left(\frac{q}{p} - q - \varepsilon\right) B_j u \leqslant \left(\frac{q}{p} - \frac{\varepsilon}{2}\right) a_j^p - A_j a_j^p - q B_j a_j.$$

Divide the inequality by a_j^p, use the asymptotic expressions for A_j, B_j and denote $v := u/a_j \geqslant 1$. We must prove that

$$\alpha(v) := \left(\frac{q}{p} - 1\right) \frac{p-1}{q-1}\, v^q + \left(\frac{q}{p} - q\right) \frac{q-p}{q-1}\, v - \frac{(p-1)(p-q)}{p}$$

$$\leqslant \beta(v) := \left(\varepsilon\, \frac{p-1}{q-1} + o(1)\right) v^q + \left(\varepsilon\, \frac{q-p}{q-1} + o(1)\right) v - \frac{\varepsilon}{2} + o(1).$$

Since the coefficient of v^q in β is positive for large j, we may bound

$$\beta(v) \geqslant \left(\varepsilon\, \frac{p-1}{q-1} + o(1)\right) v + \left(\varepsilon\, \frac{q-p}{q-1} + o(1)\right) v - \frac{\varepsilon}{2} + o(1)$$

$$= (\varepsilon + o(1))\, v - \frac{1}{2}\, \varepsilon + o(1)$$

$$\geqslant \frac{1}{2}\, \varepsilon + o(1).$$

Thus we are led to prove that $\alpha \leqslant 0$, i.e. that

$$\theta(v) := v^q - qv + q - 1 \geqslant 0, \quad \forall v \geqslant 1.$$

Now, $\theta(1) = 0$ and $\theta' \geqslant 0$, and this establishes property (A1).

Property (A2) is obvious. As for (A3), observe that condition (1) implies $A_j \geqslant 2a_j^{p-q}/3$ and $|B_j| \leqslant a_j^{p-1}/3$ for large j. Thus, for $u > a_j$ (the case $u < -a_j$ is similar),

$$g_j(u)\, u \geqslant A_j u^q - |B_j|\, u \geqslant \frac{2}{3}\, a_j^{p-q} u^q - \frac{1}{3}\, a_j^{p-1} u$$

$$= \frac{1}{3}\, a_j^{p-q} u^q + \frac{1}{3}\, a_j^p \left(\frac{u^q}{a_j^q} - \frac{u}{a_j}\right) \geqslant \frac{1}{3}\, a_j^{p-q} u^q.$$

At last, observe that the first inequality in (A4) follows readily from property (A1). In order to prove the second inequality, we show that there exist $C>0$, $j_0 \in \mathbb{N}$ such that $G_j(u) \leqslant C g_j(u)\, u$ for every $j \geqslant j_0$, $u > a_j$. Indeed, since $q>1$ and $G(a_j) \leqslant 2/p a_j^p$ for large j, this is implied by

$$\frac{2}{p} a_j^p + A_j \frac{u^q}{q} + B_j u - A_j \frac{a_j^q}{q} - B_j a_j \leqslant C \left(A_j \frac{u^q}{q} + B_j u \right).$$

Using the asymptotic expressions for A_j, B_j, we are led to prove that for C, j_0 large,

$$(C-1)\, a_j^{-p} \left(A_j \frac{u^q}{q} + B_j u \right) \geqslant \delta + o(1),$$

where $\delta := ((p-q)/pq)(p-1) + (1/p) > 0$. That is, we must prove that

$$\frac{A_j}{q} \frac{u^q}{a_j^q} + \frac{B_j}{a_j^p} u \geqslant \varepsilon$$

for some ε and every $u \geqslant a_j$. Now, the left hand member equals

$$\begin{aligned} &\left(\frac{p-1}{q-1} \frac{1}{q} + o(1) \right) \frac{u^q}{a_j^q} + \left(\frac{q-p}{q-1} + o(1) \right) \frac{u}{a_j} \\ &\quad \geqslant \left(\frac{p-1}{q-1} \frac{1}{q} + o(1) \right) \frac{u}{a_j} + \left(\frac{q-p}{q-1} + o(1) \right) \frac{u}{a_j} \\ &\quad = \left(\frac{q-p+1}{q} + o(1) \right) \frac{u}{a_j}, \end{aligned}$$

and the claim follows from condition $p-q<1$ in (1).

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